HW3 Handwritten Assignment

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Problem 1 (LSTM Cell)(0.5%)

In this exercise, we will simulate the forward pass of a simple LSTM cell. Figure.1 shows a single LSTM cell, where z is the cell input, z_i, z_f, z_o are the control inputs of the gates, c is the cell memory, and f, g, h are activation functions. Given an input x, the cell input and the control inputs can be calculated by the following equations :

- $z = w \cdot x + b$
- $z^i = w_i \cdot x + b_i$
- $z^f = w_f \cdot x + b_f$
- $z^o = w_o \cdot x + b_o$

where w, w_i, w_f, w_o are weights and b, b_i, b_f, b_o are biases. The final output can be calculated by

$$y = f(z^o)\ h(c')$$

where the value stored in cell memory is updated by

$$c' = f(z^i)g(z) + cf(z^f)$$

Note that $f(z) = \frac{1}{1+e^{-z}} \quad, g(z) = z, \quad h(z) = z$

Given an input sequence x^t $(t = 1, 2, 3, 4)$, please derive the output sequence y_t. The input sequence, the weights, and the activation functions are provided below.

$$\begin{array}{ll}
w = [0, 0, 1, 0] & , b = 0 \\
w_i = [50, 50, 0, 0] & , b_i = -5 \\
w_f = [-50, -50, 0, 0], & , b_f = 120 \\
w_o = [0, 0, 200, 0] & , b_0 = -30 \\
x^1 = [0, 0, 1, 3] & , x^2 = [0, 1, -1, 2] \\
x^3 = [2, 1, 3, 4] & , x^4 = [0, 1, 0, 0]
\end{array}$$

The initial value in cell memory is 0. **Please note that your calculation process is required to receive full credit.**

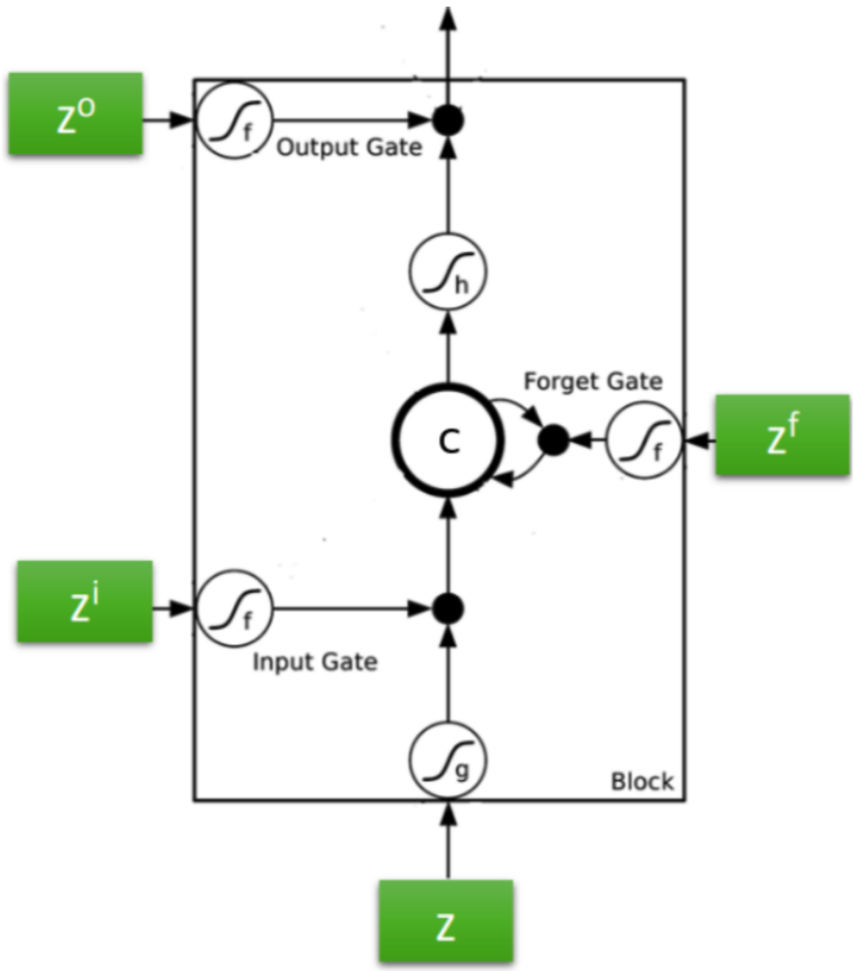


Figure 1: Problem 1 LSTM model

Problem 2 (Laplacian Eigenmaps)(1.5%)

Consider an undirected connected graph G, which is shown below. We want to utilize Laplacian Eigenmaps method to reduce these 10 points to 3-dimensional space. Here, undirected graph means that edges in the graph do not have a direction, and connected graph means that there is a path from any node to any other node in the graph.

1. Write down the adjacency matrix $\boldsymbol{W}$
2. Write down the diagonal matrix $\boldsymbol{D} = diag(d_1, ..., d_{10})$, where $d_i = \sum_{j=1}^{10} \frac{\boldsymbol{W_{ij}} + \boldsymbol{W_{ji}}}{2}$ and the Laplacian $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{W}$.
3. By HW2 Problem 3, Neighbor Embedding Slide p.7-p.10 and programming tools(MATLAB, Python...), solve the optimization problem
   minimize $\quad Trace(\boldsymbol{\Psi}^T \boldsymbol{L} \boldsymbol{\Psi})$
   subject to $\quad \boldsymbol{\Psi}^T \boldsymbol{D} \boldsymbol{\Psi} = \boldsymbol{I}_3$
   variables $\quad \boldsymbol{\Psi} \in \mathbb{R}^{10 \times 3}$
   Also, please plot the reduced points $\boldsymbol{z_1}, ..., \boldsymbol{z_{10}}$ in 3-D scatter plot.

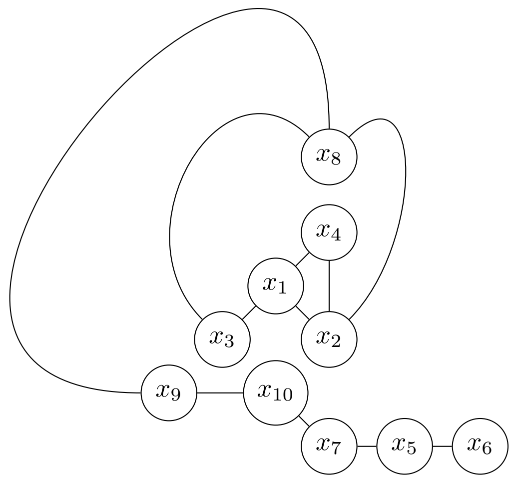


Figure 2: Problem 2 undirected connected graph G

4. You may find that the minimal eigenvalue of $\boldsymbol{L}$ is 0, and the corresponding eigenvector is

$$\begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix} \tag{1}$$

   where c is a constant. Since all the points fall into a plane, the span of these points is $\mathbb{R}^2$. In order to construct $\boldsymbol{z_1}, ..., \boldsymbol{z_{10}}$ such that span$\{\boldsymbol{z_1}, ..., \boldsymbol{z_{10}}\} = \mathbb{R}^3$, we need choose the second, third, fourth smallest eigenvalue and the corresponding eigenvectors. Please plot the reduced points by the updated $\boldsymbol{z_1}, ..., \boldsymbol{z_{10}}$ in 3-D scatter plot and verify that whether $\quad Trace(\boldsymbol{\Psi}^T \boldsymbol{L} \boldsymbol{\Psi}) = 1.098$ and $\quad \boldsymbol{\Psi}^T \boldsymbol{D} \boldsymbol{\Psi} = \boldsymbol{I}_3$.

5. Show that for no matter the graph is, there is an eigenvector of $\boldsymbol{L}$

$$\begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix} \tag{2}$$

   where c is a constant, and the corresponding eigenvalue is 0.

6. By Neighbor Embedding Slide p.9, please show that

   $\forall \boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix} \in \mathbb{R}^N, \boldsymbol{f}^T \boldsymbol{L} \boldsymbol{f} = \frac{1}{2} \sum\limits_{1 \le i,j \le N} w_{ij}(f_i - f_j)^2.$

7. Show that if $\boldsymbol{f}$ is an eigenvector of $\boldsymbol{L}$ which corresponds to eigenvalue 0, then $\boldsymbol{f}^T \boldsymbol{L} \boldsymbol{f} = 0$.

8. Show that if the graph is connected, the second smallest eigenvalue of $\boldsymbol{L}$ will be nonzero.

Problem 3 (Multiclass AdaBoost)(1.5%)

Let $\mathcal{X}$ be the input space, $\mathscr{F}$ be a collection of multiclass classifiers that map from $\mathcal{X}$ to $[1, K]$, where K denotes the number of classes. Let $\{(x_i, \hat{y}_i)\}_{i=1}^m$ be the training data set, where $x_i \in \mathcal{X}$ and $\hat{y}_i \in [1, K]$. Given $T \in \mathbb{N}$, suppose we want to find functions

$$g_{T+1}^k(x) = \sum_{t=1}^T \alpha_t f_t^k(x), \quad k \in [1, K]$$

where $f_t \in \mathscr{F}$ and $\alpha_t \in \mathbb{R}$ for all $t \in [1, T]$. Here for $f \in \mathscr{F}$, we denote $f^k(x) = \mathbf{1}\{f(x) = k\}$, where $\mathbf{1}(\cdot)$ is an indicator function, as the k'th element in the one-hot representation of $f(x) \in [1, K]$. The aggregated classifier $h : \mathcal{X} \to [1, K]$ is defined as

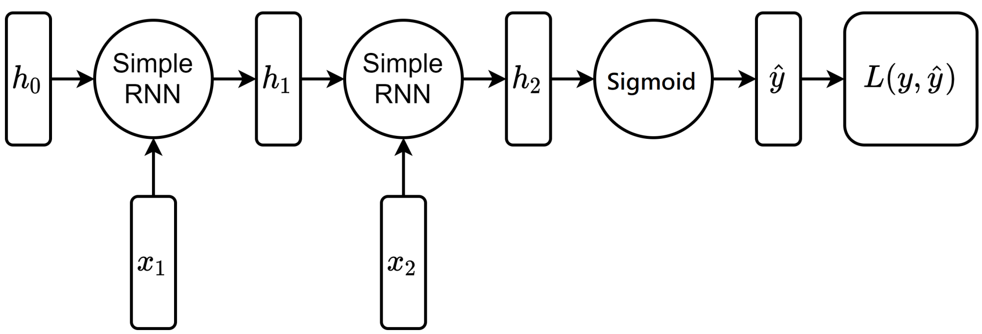
$$x \mapsto \underset{1 \leq k \leq K}{\operatorname{argmax}} \, g_{T+1}^k(x)$$

Please apply gradient boosting to show how the functions f_t and coefficients α_t are computed with an aim to minimize the following loss function

$$L((g_{T+1}^1, \cdots, g_{T+1}^K) = \sum_{i=1}^m \exp\left(\frac{1}{K-1} \sum_{k \neq \hat{y}_i} g_{T+1}^k(x_i) - g_{T+1}^{\hat{y}_i}(x_i)\right)$$

Problem 4 (Backpropagation through time via Simple RNN)(1%)

Backpropagation through time is a critical concept to know as we train a recurrent network. Here, we set a toy case of prediction problem. The Simple RNN module has two kinds of weights, w_x and w_h, such that



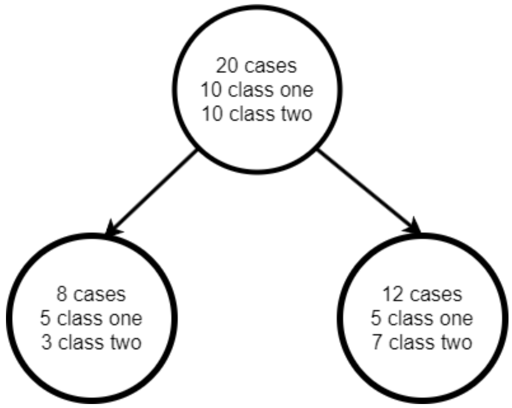
$h_t = \tanh(w_x x_t + w_h h_{t-1})$, where t represents the index of steps. The module has the weight w_o such that $\hat{y} = \sigma(w_o h_2)$, where $\sigma(w_o h_2) = \frac{1}{1+\exp(-w_o h_2)}$. The initial state h_0 is set to be 0. The sequential input only contains $\{x_1, x_2\}$; the label is y; the loss function is MSE. Please derive $\frac{\partial L(y,\hat{y})}{\partial w_o}$, $\frac{\partial L(y,\hat{y})}{\partial w_h}$, $\frac{\partial L(y,\hat{y})}{\partial w_x}$ in terms of x_1, x_2, h_0, h_1, h_2, w_x, w_o, and w_h.

Problem 5 (Loss function of Decision tree) (1.5%)

It is known that decision tree is still a powerful classification model now a days. There are two different loss functions when it comes to entropy counting, which are Shannon information gain and Gini index. Following are their definition:

$$\text{Gini index} = \frac{N_{left}}{N}\left(1 - \sum_{i=1}^c \left(p_{left}^i\right)^2\right) + \frac{N_{right}}{N}\left(1 - \sum_{i=1}^c \left(p_{right}^i\right)^2\right)$$

$$\text{Shannon information gain} = \frac{N_{left}}{N}\left(-\sum_{i=1}^c p_{left}^i \log_2 p_{left}^i\right) + \frac{N_{right}}{N}\left(-\sum_{i=1}^c p_{right}^i \log_2 p_{right}^i\right)$$



$p_i^j :=$ the proportion of class j in the node i.
$N_i :=$ the number of cases in the node i.
Now we give a toy example. In this case $N_{left} = 8, p^1_{left} = \frac{5}{8}, p^2_{left} = \frac{3}{8}, N_{right} = 12, p^1_{right} = \frac{5}{12}, p^2_{right} = \frac{7}{12}$

In the following questions we consider classification of two cases. Please calculate the entropy of the following questions using two above-mentioned loss functions.

(a) (i) A 50/50 split with the first part containing 80% of positive examples and the second part containing 75% of positive examples.

(ii) A 80/20 split with the first part containing 0% of positive examples and the second part containing 90% of positive examples.

(iii) A 90/10 split with the first part containing 1% of positive examples and the second part containing 100% of positive examples.

(b) However, now suppose that our case is to detect the covid-19. Thus, we want our entropy function can have a higher loss on (iii) than (ii).Please decide a function that fulfill this criteria and write down the loss.

Version Description

1. First Edition: Finish Problem 1 to 5
2. Second Edition: Updated Problem 2(4) 2(5) 2(6) 2(7)'s typo